

The Self-Amalgamation of Coronas and Generalized Crowns*

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Abstract

Let $G = (V(G), E(G))$ be a finite, connected, simple graph. Let u and v be two vertices of G such that the distance between u and v is at least 3. A ***self-amalgamation*** of G , denoted by $G*$ with $*$ $= (u, v)$, is the graph obtained by identifying u and v . A k^{th} ***self-amalgamation*** of G , denoted by $G*^k$, is a self-amalgamation of a $(k-1)^{\text{st}}$ self-amalgamation of G , that is, $G*^k = (G*^{k-1})*$. A graph G is ***self-amalgamation stable*** (or ***amalgamation-stable***) if a $G*$ is not possible. If G is not amalgamation-stable, then the ***stability number*** of G is the minimum positive integer k such that there exists a $G*^k$ which is amalgamation-stable. The ***self-amalgamation number*** of G , denoted by $s(G)$, is the minimum positive integer k such that all the k^{th} self-amalgamations of G are amalgamation-stable. Results on the stability number and self-amalgamation number of coronas and generalized crowns are presented.

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1. Introduction

This study considers only graphs which are simple, finite and connected, unless otherwise stated. Graph-theoretic terms which are used but not explicitly defined in this study are adopted from [1] and [4]. Related investigations on the k -amalgamation of graphs may be found in [2], [3] and [5]. This study is on the stability number and self-amalgamation number of coronas and generalized crowns.

Let $G = (V(G), E(G))$ be a simple, finite connected graph. The **order** of G , denoted by $|V(G)|$, is the number of vertices in G . The **size** of G , denoted by $\varepsilon(G)$, is the number of edges in G . The **distance** between two vertices u and v of G , denoted by $d(u, v; G)$, is the length of a shortest path in G between u and v . Whenever only one graph is under consideration, the distance between u and v will be denoted by $d(u, v)$. The **diameter** of G , denoted by $diam(G)$, is the maximum distance between any two vertices of G .

Definition 1.1. Let $G = (V(G), E(G))$ be a finite, connected, simple graph. Let u and v be two vertices of G such that the distance between u and v is at least 3, that is $d(u, v; G) \geq 3$. A **1st self-amalgamation** (or **self-amalgamation**) of G , denoted by G^* , with $*$ = (u, v) or $G^*(u, v)$, is the graph obtained by identifying the vertices u and v .

Definition 1.2. ([2], [4]) For $k \geq 2$, a **k^{th} self-amalgamation of G** , denoted by G^{*^k} , is defined recursively as a self-amalgamation of a $(k-1)^{\text{st}}$ self-amalgamation of G , that is, $G^{*^k} = (G^{*^{(k-1)}})^*$. A **k^{th} self-amalgamation of G** which is obtained with $*^k = (\langle x_i \rangle_{i=1}^k, \langle y_i \rangle_{i=1}^k)$, where $\langle x_i \rangle$ and $\langle y_i \rangle$ are sequences of vertices in G such that x_i is identified with y_i for each $i = 1, 2, 3, \dots, k$ may be denoted by $G^*(\langle x_i \rangle_{i=1}^k, \langle y_i \rangle_{i=1}^k)$. The sequences $\langle x_i \rangle$ and $\langle y_i \rangle$ are called **amalgamation sequences**.

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The k^{th} self-amalgamation $G *^k$ with $*^k = (\langle x_i \rangle_{i=1}^k, \langle y_i \rangle_{i=1}^k)$ may also be denoted by the following sequence of k self-amalgamations of G :

$$((((G * (x_1, y_1)) * (x_2, y_2)) * (x_3, y_3)) * \dots * (x_k, y_k)).$$

Example 1.1. A 1st self-amalgamation (or self-amalgamation) of a path $P_8 = (1, 2, 3, 4, 5, 6, 7, 8)$, with $* = (2, 6)$, is shown in Figure 1.1(a). Another self-amalgamation of P_8 with $* = (2, 8)$ is shown in Figure 1.1(b).

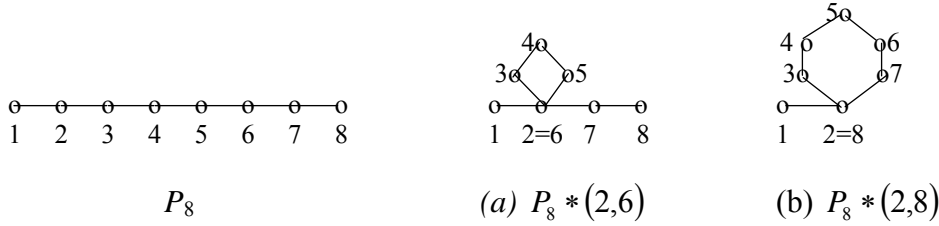


Figure 1.1 Some non-isomorphic 1st self-amalgamations of P_8

Note that the 1st self-amalgamations $P_8 * (2, 6)$ and $P_8 * (2, 8)$ given in Example 1.1 are not isomorphic. In general, different amalgamation sequences may yield non-isomorphic self-amalgamations.

Example 1.2. The 2nd self-amalgamation $P_8 *^2$ with $*^2 = (\langle 1, 1 \rangle, \langle 4, 7 \rangle)$ is obtained through the following sequence of two self-amalgamations. Step 1. Use $* = (1, 4)$ to get $P_8 * (1, 4)$. Step 2. Apply the second amalgamation $* = (1, 7)$ to get $(P_8 * (1, 4)) * (1, 7) = P_8 * (\langle 1, 1 \rangle, \langle 4, 7 \rangle)$. (Refer to Figure 1.2.)

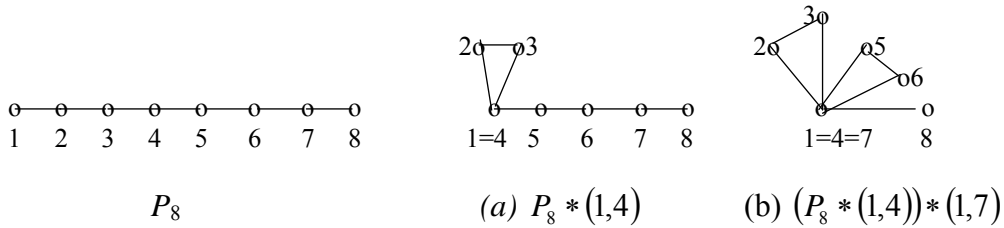


Figure 1.2 A 2nd self-amalgamation of P_8 with $*^2 = (\langle 1, 1 \rangle, \langle 4, 7 \rangle)$

Definition 1.3. A graph G is said to be *self-amalgamation stable* (or *amalgamation-stable*) if a $G *$ is not possible. A graph G is *amalgamation-unstable* if G is not self-amalgamation stable; in this case we say that a $G *$ is possible.

Remark 1.1. G is amalgamation-stable whenever G^* is not possible. Thus, a graph G is self-amalgamation-stable (or amalgamation-stable) whenever $d(u, v; G) \leq 2$ for any vertices u and v in G . Equivalently, a graph G is amalgamation-stable if and only if $\text{diam}(G) \leq 2$.

Remark 1.2. If G^{*^k} is possible, then the order of G^{*^k} is $|V(G^{*^k})| = |V(G)| - k$ and the size is $\varepsilon(G^{*^k}) = \varepsilon(G)$.

Example 1.3. The graphs $P_8^*(2,6)$ and $P_8^*(2,8)$ in Figure 1.1 are amalgamation-unstable since there are vertices u and v in these graphs with $d(u, v) \geq 3$. For example, $d(3, 8; P_8^*(2,6)) = 3$ and $d(5, 8; P_8^*(2,8)) = 3$. Figure 1.2 (b) shows the graph $P_8^{*^2}$ with $*^2 = (\langle 1, 1 \rangle, \langle 4, 7 \rangle)$, which is amalgamation-stable.

Example 1.4. The diameter of the Petersen graph is 2; hence, the Petersen graph is amalgamation-stable. The cycles C_n with $n = 3, 4, 5$ are also amalgamation-stable.

The following result gives a necessary condition for the existence of a G^{*^k} .

Lemma 1.1. If G^{*^k} is possible, then $|V(G)| \geq k + 3$.

Proof. If a G^{*^k} is possible, then there is a $G^{*^{k-1}}$ which is amalgamation-unstable. Thus, there exist vertices u and v in $G^{*^{k-1}}$ such that $d(u, v; G^{*^{k-1}}) \geq 3$. Thus, $G^{*^{k-1}}$ contains a u - v path of order 3, and hence, $|V(G^{*^{k-1}})| \geq 4$. Therefore, $|V(G^{*^{k-1}})| = |V(G)| - (k-1) \geq 4$ and the desired result follows. ■

Remark 1.3. If $k > |V(G)| - 3$, then a G^{*^k} is not possible.

Definition 1.4. Let G be an amalgamation-unstable graph, that is, $\text{diam}(G) \geq 3$. For $m \geq 1$, the m^{th} *amalgamation set*, denoted by $S^m(G)$, is the set of all the non-isomorphic m^{th} self-amalgamations of G together with all the non-isomorphic stable k^{th} self-amalgamations of G for each $1 \leq k < m$. When G is amalgamation-stable, we define $S^0(G) = \{G\}$.

Definition 1.5. Let G be an amalgamation-unstable graph, that is, $\text{diam}(G) \geq 3$. The *stability number* of G , denoted by $*(G)$, is the minimum positive integer k such that there is a G^{*^k} which is amalgamation-stable. The *self-amalgamation number* of G , denoted by $s(G)$, is the minimum integer k such that all the graphs in $S^k(G)$ are amalgamation-stable. When G is amalgamation-stable, we define $*(G) = 0$ and $s(G) = 0$.

Remark 1.4. The stability number of a graph G is the minimum k such that $S^k(G)$ contains an amalgamation-stable graph.

The next example illustrates the definitions of m^{th} -amalgamation set, stability number and self-amalgamation number for P_6 .

Example 1.5. Consider the path $P_6 = (1, 2, 3, 4, 5, 6)$. Since $\text{diam}(P_6) = 5$, so P_6 is amalgamation-unstable. Thus, the stability number of P_6 is not 0. The 1st amalgamation set $S^1(P_6)$ consists of the non-isomorphic self-amalgamations $P_6 * (u, v)$, where u and v are vertices in P_6 such that $d(u, v; P_6) \geq 3$. Thus, we have the set $S^1(P_6) = \{P_6 * (1, 6), P_6 * (2, 5), P_6 * (1, 4), P_6 * (1, 5)\}$. (Refer to Figure 1.3.) We observe that $\text{diam}(P_6 * (1, 6)) = 2$ and $\text{diam}(P_6 * (2, 5)) = 2$; hence, $P_6 * (1, 6)$ and $P_6 * (2, 5)$ are both amalgamation-stable. Therefore, the stability number of P_6 is $s(P_6) = 1$. However, $d(2, 6; P_6 * (1, 4)) = 3$, and $d(3, 6; P_6 * (1, 5)) = 3$. Hence, both $P_6 * (1, 4)$ and $P_6 * (1, 5)$ are amalgamation-unstable, and the self-amalgamation number $s(P_6)$ of P_6 is greater than 1. The 2nd amalgamation set $S^2(P_6)$ consists of the amalgamation-stable graphs in $S^1(P_6)$, together with the non-isomorphic self-amalgamations of $P_6 * (1, 4)$ and $P_6 * (1, 5)$. The 2nd self-amalgamations of P_6 are represented by $P_6 * (\langle 1, 2 \rangle, \langle 4, 6 \rangle) = (P_6 * (1, 4)) * (2, 6)$ and $P_6 * (\langle 1, 3 \rangle, \langle 5, 6 \rangle) = (P_6 * (1, 5)) * (3, 6)$, which are isomorphic graphs with diameter 2. Therefore, the 2nd-amalgamation set is $S^2(P_6) = \{P_6 * (1, 6), P_6 * (2, 5), P_6 * (\langle 1, 3 \rangle, \langle 5, 6 \rangle)\}$. All the graphs in $S^2(P_6)$ are amalgamation-stable. Therefore, the self-amalgamation number of P_6 is $s(P_6) = 2$.

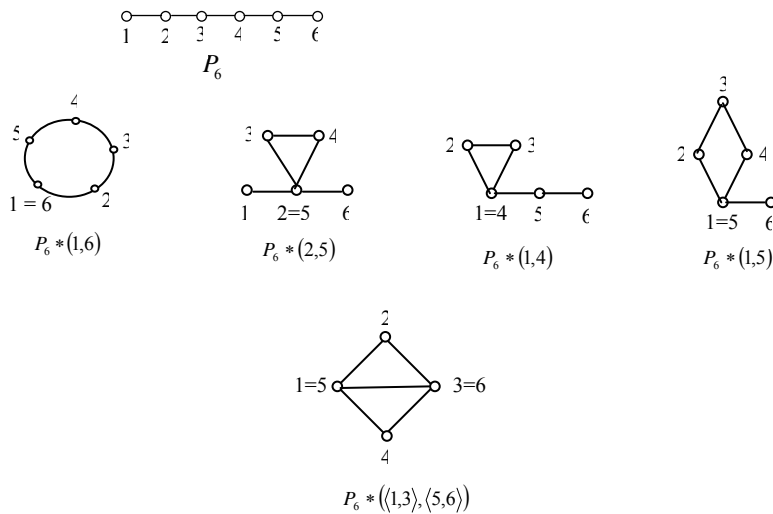


Figure 1.3 The non-isomorphic 1st and 2nd self-amalgamations of P_6

We will next consider a class of amalgamation-stable graphs.

Definition 1.6. Let G and H be connected graphs with disjoint vertex sets. The **sum** or **join** of G and H , denoted by $G + H$, is the graph whose vertex set is $V(G) \cup V(H)$ and whose edge set is $E(G) \cup E(H)$ together with all edges joining each vertex in $V(G)$ with each vertex in $V(H)$.

Example 1.6. The sum of a cycle $C_5 = [1, 2, 3, 4, 5]$ and a path $P_3 = (a, b, c)$ is shown in Figure 1.4.

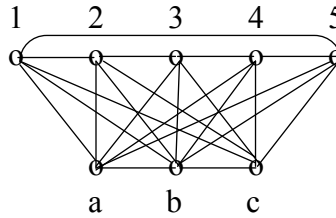


Figure 1.4 The sum $C_5 + P_3$

Theorem 1.2. For any graphs G and H , $\text{diam}(G + H) = 1$ if and only if G and H are complete graphs.

Proof. This follows from the definition of the sum of two graphs.

Theorem 1.3. Let G and H be graphs, each of which may be disconnected. If G or H is not a complete graph, then $\text{diam}(G + H) = 2$.

Proof. Suppose G or H has nonadjacent vertices. Without loss of generality, let u and v be nonadjacent vertices in G . For any vertex w in H , the vertex pairs u, w and v, w are adjacent in $G + H$, thus the path $P_3 = (u, w, v)$ exists in $G + H$. Hence, $d(u, v; G + H) = 2$ for any two nonadjacent vertices u and v of G , and $d(u, v; G + H) = 1$ if u and v are adjacent in G , or u is in G and v is in H . Therefore, $\text{diam}(G + H) = 2$ if G or H is not a complete graph. ■

Corollary 1.4. Let G and H , each of which may be disconnected. The sum $G + H$ is amalgamation stable, with stability number $*(G + H) = 0$ and self-amalgamation number $s(G) = 0$.

By Corollary 1.4, the following classes of graphs, which can be obtained as a sum of graphs, have stability number $*(G) = 0$ and self-amalgamation number $s(G) = 0$.

- i. The wheels $W_n = C_n + K_1$ and the fans $F_n = P_n + K_1$
- ii. The complete bipartite graphs $K_{m,n} = \overline{K_m} + \overline{K_n}$

iii. The complete r -partite graphs $K_{m_1, m_2, m_3, \dots, m_r} = \overline{K_{m_1}} + \overline{K_{m_2}} + \dots + \overline{K_{m_r}}$

2. The Self-Amalgamation of Coronas and Crowns

Definition 2.1. Let G and H be connected graphs. Let the number of vertices of G be $|V(G)| = n$, and let the vertices of G be labeled $1, 2, 3, \dots, n$. A **corona** $G \circ H$ is the graph obtained by making n copies of H and connecting every vertex of the i^{th} copy of H to vertex i of G . A **crown** is the corona $C_n \circ P_1$, where $n \geq 3$ and P_1 is the degenerate path of order 1. A **generalized crown** is the corona $C_n \circ P_m$, $n \geq 3$, $m \geq 2$.

In this section, a cycle of order n will be denoted by $C_n = [1, 2, 3, \dots, n]$, a path of order m by $P_m = (a_1, a_2, a_3, \dots, a_m)$, and the i^{th} copy of P_m by $P_m^i = (a_1^i, a_2^i, a_3^i, \dots, a_m^i)$.

Example 2.1. The crown $C_4 \circ P_1$, the generalized crown $C_4 \circ P_2$, and the corona $P_3 \circ P_2$ are shown in Figure 2.1.

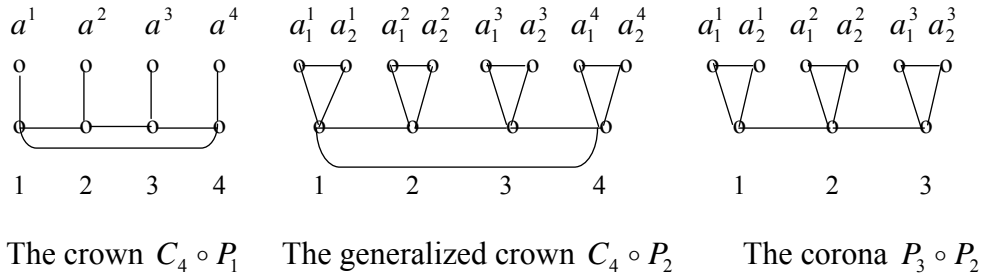


Figure 2.1 Some examples of coronas

Theorem 2.1. $\text{diam}(G \circ H) = \text{diam}(G) + 2$ if G is a connected graph with at least two vertices and H is any graph.

Proof. The result follows from the definition of a corona. ■

The next results follow from Theorem 2.1 and the fact that $\text{diam}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$ for any cycle C_n , $n \geq 3$.

Corollary 2.2. For $n \geq 3$ and any graph H , the diameter of the corona $C_n \circ H$ is

$$\text{diam}(C_n \circ H) = \left\lfloor \frac{n}{2} \right\rfloor + 2.$$

Corollary 2.3. Let G and H be connected graphs. Then

(i) If G is the trivial graph, then $*(G \circ H) = 0$ and $s(G \circ H) = 0$.

(ii) If G has at least two vertices, then $*(G \circ H) \geq 1$ and $s(G \circ H) \geq 1$.

Proof. The result follows from Theorem 2.1 and the observation that the diameter of a connected graph G is at least 1, unless $G = P_1$. ■

Corollary 2.4. Let G be a connected graph. A corona $G \circ H$ is amalgamation-stable if and only if $|V(G)| = 1$.

Let the crown $C_n \circ P_1$ be obtained by joining the vertex a^i of the i^{th} copy of P_1 to vertex i of the cycle $C_n = [1, 2, 3, \dots, n]$. The next results establish bounds on the stability number and the self-amalgamation number of crowns. We first consider the crown $C_3 \circ P_1$.

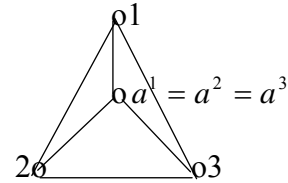
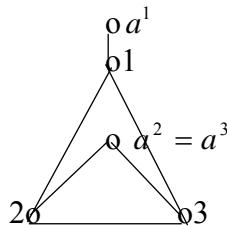
Theorem 2.5. The 1st and 2nd amalgamation sets of $C_3 \circ P_1$ are:

(i) $S^1(C_3 \circ P_1) = \{(C_3 \circ P_1) * (a^2, a^3)\}$

(ii) $S^2(C_3 \circ P_1) = \{W_3\}$

Moreover, $(C_3 \circ P_1) * (a^2, a^3)$ is amalgamation-unstable.

Proof. (i) We note that $d(u, v; C_3 \circ P_1) = 3$ if and only if $u = a^i, v = a^j$, where $i \neq j$ and $i, j = 1, 2, 3$. Thus, any self-amalgamation will involve only the identification of the vertices of two copies of P_1 . Without loss of generality, consider $* = (a^2, a^3)$ and the resulting 1st self-amalgamation; all other 1st self-amalgamations are isomorphic to this graph (refer to Figure 2.2(a)). Thus, $S^1(C_3 \circ P_1) = \{(C_3 \circ P_1) * (a^2, a^3)\}$. Since $d(a^1, a^3; (C_3 \circ P_1) *) = 3$, so $(C_3 \circ P_1) * (a^2, a^3)$ is amalgamation-unstable. Any 2nd self-amalgamation is isomorphic to $(C_3 \circ P_1) *^2$, with $*^2 = (\langle a^2, a^2 \rangle, \langle a^3, a^1 \rangle)$. This is equivalent to $((C_3 \circ P_1) * (a^2, a^3)) * (a^2, a^1)$, which is the wheel W_3 and is amalgamation-stable (refer to Figure 2.2(b)). Thus, $S^2(C_3 \circ P_1) = \{W_3\}$. ■



(a) $S^1(C_3 \circ P_1) = \{(C_3 \circ P_1) * \}, \text{ where } * = (a^2, a^3)$ (b) $S^2(C_3 \circ P_1) = \{W_3\}$

Figure 2.2 $S^n(C_3 \circ P_1)$, $n = 1, 2$

Corollary 2.6. 4.3. $*(C_3 \circ P_1) = 2$ and $s(C_3 \circ P_1) = 2$.

Proof. The result follows from Theorem 2.5. ■

Theorem 2.7. For $n \geq 4$, $1 \leq *(C_n \circ P_1) \leq n - 1$.

Proof. Consider the $(n-1)^{\text{st}}$ self-amalgamation of $C_n \circ P_1$ defined by $*^{n-1} = (X, Y)$, with $X = \langle x_i \rangle_1^{n-1} = \langle a^1, a^1, a^1, \dots, a^1 \rangle$ and $Y = \langle y_i \rangle_1^{n-1} = \langle a^2, a^3, \dots, a^n \rangle$. Then, $(C_n \circ P_1) *^{n-1}$ is isomorphic to the wheel W_n , $n \geq 3$, which is amalgamation-stable. Therefore, $*(C_n \circ P_1) \leq n - 1$, $n \geq 3$. By Corollary 2.3, $*(C_n \circ P_1) \geq 1$. This completes the proof. ■

Theorem 2.8. Let G be a connected graph with $\text{diam}(G) \leq 2$ and order $|V(G)| = n \geq 2$.

Then, $1 \leq *(G \circ P_1) \leq n - 1$.

Proof. The $(n-1)^{\text{st}}$ self-amalgamation of $G \circ P_1$ defined by $*^{n-1} = (X, Y)$, with $X = \langle x_i \rangle_1^{n-1} = \langle a^1, a^1, a^1, \dots, a^1 \rangle$ and $Y = \langle y_i \rangle_1^{n-1} = \langle a^2, a^3, \dots, a^n \rangle$ has diameter 2. Thus, $(G \circ P_1) *^{n-1}$ is amalgamation-stable. Therefore, $1 \leq *(G \circ P_1) \leq n - 1$. ■

Example 2.2. Consider the 6^{th} self-amalgamation of $C_7 \circ P_1$ defined by $*^6 = (X, Y)$, with $X = \langle x_i \rangle_1^6 = \langle a^1, a^1, a^1, \dots, a^1 \rangle$ and $Y = \langle y_i \rangle_1^6 = \langle a^2, a^3, a^4, a^5, a^6, a^7 \rangle$ shown in Figure 2.3. Note that for this choice of amalgamation sequences, $(C_7 \circ P_1) *^6$ is isomorphic to the wheel W_7 , which is amalgamation-stable. Thus, $*(C_7 \circ P_1) \leq 6$.

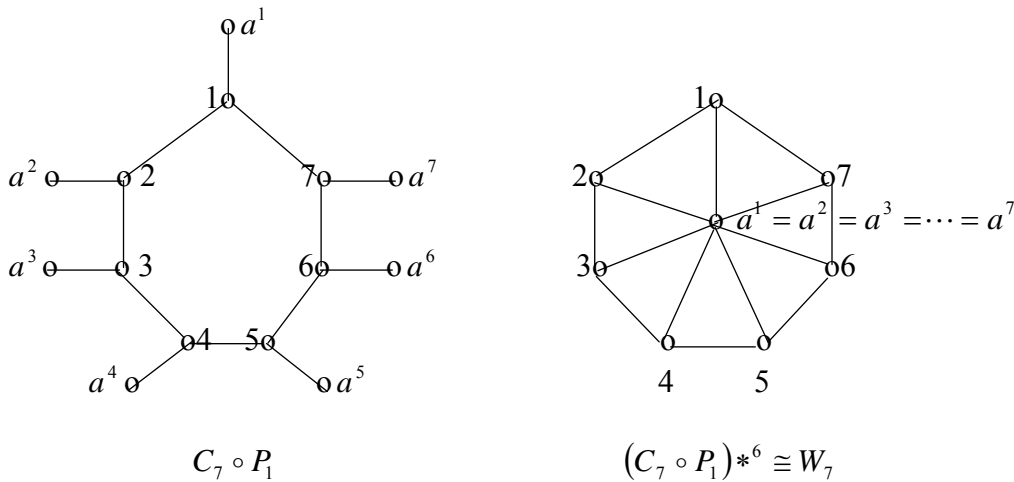


Figure 2.3 An amalgamation-stable 6^{th} self-amalgamation of $C_7 \circ P_1$,
with $*^6 = (\langle a^1, a^1, a^1, \dots, a^1 \rangle, \langle a^2, a^3, a^4, a^5, a^6, a^7 \rangle)$

Theorem 2.9. For $n \geq 7$, $n \leq s(C_n \circ P_1) \leq 2n - 3$.

Proof. By Remark 1.3, $s(C_n \circ P_1) \leq 2n - 3$. Consider the $(n-1)^{st}$ self-amalgamation of $C_n \circ P_1$ defined by $*^{n-1} = (X, Y)$, with $X = \langle x_i \rangle_1^{n-1} = \langle a^1, a^2, a^2, \dots, a^2, a^n \rangle$ and $Y = \langle y_i \rangle_1^{n-1} = \langle 3, a^3, a^4, \dots, a^{n-1}, 2 \rangle$. A shortest path between 1 and $n-2$ is the path $P_4 = (1, n, n-1, n-2)$, where $n-2 \geq 5$; hence, $d(1, n-2; (C_n \circ P_1)^{*^{n-1}}) = 3$. Thus $(C_n \circ P_1)^{*^{n-1}}$ is amalgamation-unstable for this choice of amalgamation sequences. Hence, $S^{n-1}(G)$ contains an amalgamation-unstable graph, and therefore $s(C_n \circ P_1) \geq n$ for $n \geq 7$. ■

Example 2.3. A 6th self-amalgamation of $C_7 \circ P_1$ defined by $*^6 = (X, Y)$, with $X = \langle x_i \rangle_1^5 = \langle a^1, a^2, a^2, a^2, a^2, a^7 \rangle$ and $Y = \langle y_i \rangle_1^5 = \langle 3, a^3, a^4, a^5, a^6, 2 \rangle$. is shown in Figure 2.4. For this choice of amalgamation sequences, $(C_7 \circ P_1)^{*^6}$ is amalgamation-unstable since $d(1, 5, C_7 \circ P_1) = 3$. Thus, the self-amalgamation number of $C_7 \circ P_1$ is $s(C_7 \circ P_1) \geq 7$.

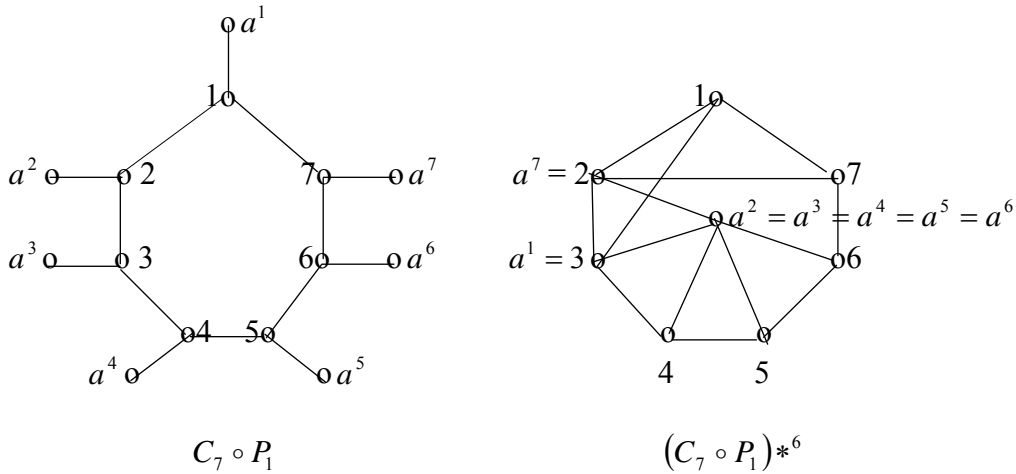


Figure 2.4 An amalgamation-unstable 6th self-amalgamation of $C_7 \circ P_1$, with $*^6 = (\langle a^1, a^2, a^2, a^2, a^2, a^7 \rangle, \langle 3, a^3, a^4, a^5, a^6, 2 \rangle)$

3. Self-Amalgamation of Generalized Crowns

We establish upper bounds on the stability number and lower bounds for the self-amalgamation number of the generalized crown $C_n \circ P_m$, $n \geq 3$, $m \geq 2$. Let $C_n = [1, 2, 3, \dots, n]$. Let a path of order m be denoted by $P_m = (a_1, a_2, a_3, \dots, a_m)$ and

the i^{th} copy of P_m by $P_m^i = (a_1^i, a_2^i, a_3^i, \dots, a_m^i)$ where each vertex of P_m^i is joined to vertex i of the cycle C_n .

We consider the stability number and the self-amalgamation number of $C_3 \circ P_2$, $n \geq 3$.

Theorem 3.1.

- (i) $S^1(C_3 \circ P_2) = \{(C_3 \circ P_2)^*\}$, where $* = (a_1^1, a_2^2)$.
- (ii) $S^2(C_3 \circ P_2) = \{G_1, G_2\}$, where $G_1 = (C_3 \circ P_2)^{*^2}$, with $*^2 = (\langle a_1^1, a_1^2 \rangle, \langle a_2^2, a_2^3 \rangle)$, and $G_2 = (C_3 \circ P_2)^{*^2}$, with $*^2 = (\langle a_1^1, a_1^1 \rangle, \langle a_1^2, a_1^3 \rangle)$.
- (iii) $S^3(C_3 \circ P_2) = \{H, G_2\}$, where $G_2 = (C_3 \circ P_2)^{*^2}$, with $*^2 = (\langle a_1^1, a_1^1 \rangle, \langle a_1^2, a_1^3 \rangle)$, and $H = (C_3 \circ P_2)^{*^3} = G_1 * (a_1^3, a_2^1)$.

Proof. (i) We note that $d(u, v; C_3 \circ P_2) \geq 3$ if and only if $u = a_m^i, v = a_n^j$, where $i \neq j$, $i, j = 1, 2, 3$, and $m, n = 1, 2$. Thus, any self-amalgamation will involve the identification of the vertices of different copies of P_2 . For $i = 1, 2, 3$, $d(a_1^i, a_2^i; C_3 \circ P_2) = d(a_1^i, a_2^i; P_2) = 1$, hence, any k^{th} self-amalgamation must not identify two vertices of a copy of P_2 . Without loss of generality, let $* = (a_1^1, a_2^2)$. The resulting graph $(C_3 \circ P_2)^*(a_1^1, a_2^2)$ is shown in Figure 3.1(a); all other 1st self-amalgamations of $C_3 \circ P_2$ are isomorphic to this graph. Thus, $S^1(C_3 \circ P_2) = \{(C_3 \circ P_2)^*\}$, where $* = (a_1^1, a_2^2)$. Note that $\text{diam}((C_3 \circ P_2)^*) = 3$ so that $(C_3 \circ P_2)^*$ is amalgamation-unstable.

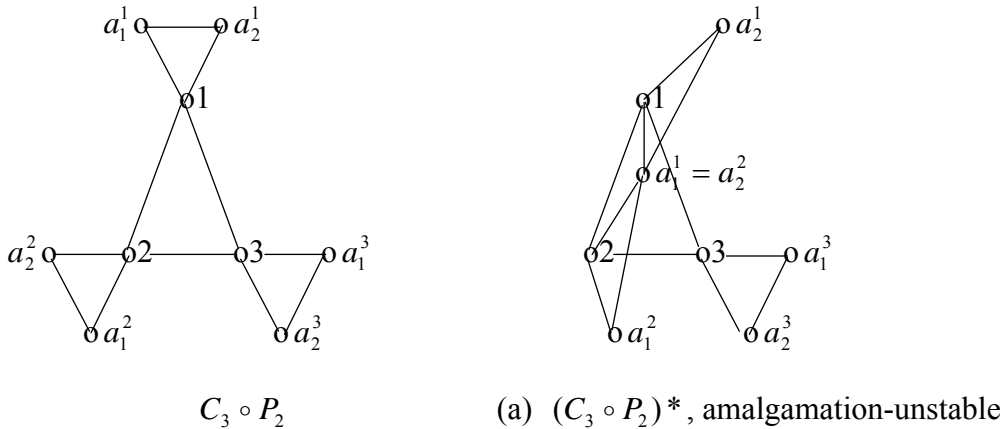
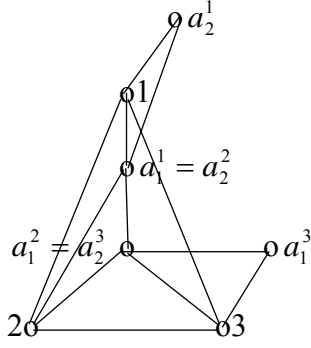


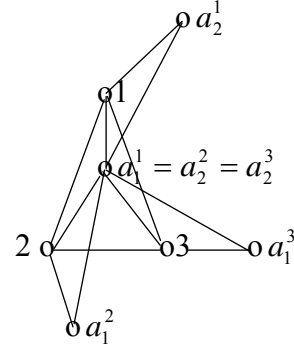
Figure 3.1 $S^1(C_3 \circ P_2) = \{(C_3 \circ P_2)^*\}$, $* = (a_1^1, a_2^2)$

- (ii) The vertices which are at distance 3 in the 1st self-amalgamation obtained in (i) are the pairs a_1^1 and a_1^3 or a_2^3 , the pairs a_1^2 and a_1^3 or a_2^3 , and the pairs a_2^1 and a_1^3 or

a_2^3 . The non-isomorphic 2^{nd} self-amalgamations of $C_3 \circ P_2$ are G_1 and G_2 which are obtained using $*^2 = (\langle a_1^1, a_1^2 \rangle, \langle a_2^2, a_2^3 \rangle)$ and $*^2 = (\langle a_1^1, a_1^1 \rangle, \langle a_2^2, a_2^3 \rangle)$, respectively. (Refer to Figure 3.2.) The graph G_1 is amalgamation-unstable while G_2 is amalgamation-stable.

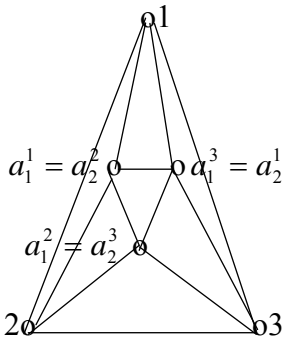


(a) $G_1 = (C_3 \circ P_2)^{*^2}$, with
 $*^2 = (\langle a_1^1, a_1^2 \rangle, \langle a_2^2, a_2^3 \rangle)$
 G_1 amalgamation-unstable

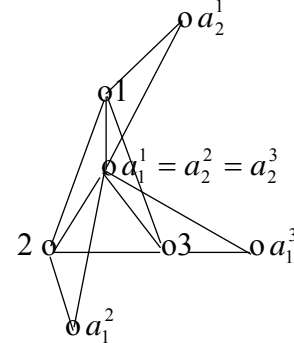


(b) $G_2 = (C_3 \circ P_2)^{*^2}$, with
 $*^2 = (\langle a_1^1, a_1^1 \rangle, \langle a_2^2, a_2^3 \rangle)$
 G_2 amalgamation-stable.

Figure 3.2 $S^2(C_3 \circ P_2) = \{G_1, G_2\}$



(a) $H = (C_3 \circ P_2)^{*^3}$, with
 $*^3 = (\langle a_1^1, a_1^2, a_1^3 \rangle, \langle a_2^2, a_2^3, a_2^1 \rangle)$
 H amalgamation-stable



(b) $G_2 = (C_3 \circ P_2)^{*^2}$, with
 $*^2 = (\langle a_1^1, a_1^1 \rangle, \langle a_2^2, a_2^3 \rangle)$
 G_2 amalgamation-stable.

Figure 3.3 $S^3(C_3 \circ P_2) = \{H, G_2\}$

(iii) G_2 is amalgamation-stable, so $G_2 \in S^3(C_3 \circ P_2)$; however $G_1 = (C_3 \circ P_2)^{*^2}$ with $*^2 = (\langle a_1^1, a_1^2 \rangle, \langle a_2^2, a_2^3 \rangle)$, is amalgamation-unstable. Note that $d(u, v; G_1) = 3$ if and only if $u = a_1^3$ and $v = a_2^1$. Thus, $H = G_1 * (a_2^1, a_1^3)$, or equivalently, $(C_3 \circ P_3)^{*^3}$ with $*^3 = (\langle a_1^1, a_1^2, a_1^3 \rangle, \langle a_2^2, a_2^3, a_2^1 \rangle)$ is amalgamation-stable. (Refer to Figure 3.3.) Thus, $S^3(C_3 \circ P_2) = \{H, G_2\}$. ■

Corollary 3.2. $*(C_3 \circ P_2) = 2$ and $s(C_3 \circ P_2) = 3$

Proof. This follows from Theorem 3.1. ■

We will consider the self-amalgamation of $C_n \circ P_m$ for other cases.

Theorem 3.3. For $n \geq 4$, $*(C_n \circ P_2) \leq n - 1$.

Proof. Let $n \geq 4$. We note that $d(u, v, C_n \circ P_2) \geq 3$ if and only if u and v are vertices of different copies of P_2 . (Refer to Figure 3.4.) Consider the $(n-1)^{st}$ self-amalgamation of $C_n \circ P_2$ defined by $*^{n-1} = (X, Y)$, with $X = \langle x_i \rangle_1^{n-1} = \langle a_1^1, a_1^1, a_1^1, \dots, a_1^1 \rangle$ and $Y = \langle y_i \rangle_1^{n-1} = \langle a_2^2, a_2^3, a_2^4, \dots, a_2^n \rangle$. Since $\text{diam}((C_n \circ P_2)^{*^{n-1}}) = 2$, so $(C_n \circ P_2)^{*^{n-1}}$ is amalgamation-stable for this choice of amalgamation sequences. Therefore, $*(C_n \circ P_2) \leq n - 1$. ■

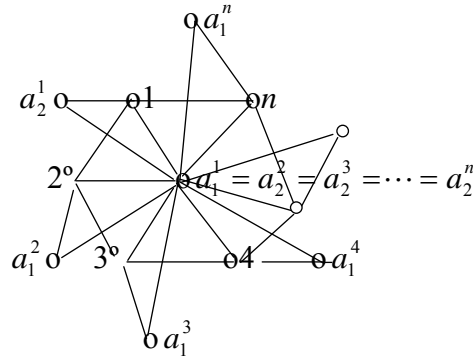


Figure 3.4 An amalgamation-stable $(C_n \circ P_2)^{*^{n-1}}$, $n \geq 4$, with

$$*^{n-1} = (\langle a_1^1, a_1^1, a_1^1, a_1^1, \dots, a_1^1 \rangle, \langle a_2^2, a_2^3, a_2^4, a_2^5, \dots, a_2^n \rangle)$$

Remark 3.1. It can be verified that all the elements of $S^1(C_4 \circ P_2)$ and $S^2(C_4 \circ P_2)$ are amalgamation-unstable, hence $*(C_4 \circ P_2) > 2$. By Theorem 3.3, $*(C_4 \circ P_2) \leq 3$. Therefore, $*(C_4 \circ P_2) = 3$.

Theorem 3.4. $s(C_4 \circ P_2) \geq 4$

Proof. In the 3rd self-amalgamation $(C_4 \circ P_2)^{*^3}$, with $*^3 = (\langle a_1^1, a_1^2, a_1^3 \rangle, \langle a_2^2, a_2^3, a_2^4 \rangle)$, the shortest path between the vertices a_2^1 and a_1^4 is $(a_2^1, 1, 4, a_1^4)$. Thus, $(C_4 \circ P_2)^{*^3}$ is amalgamation-unstable. Therefore $s(C_4 \circ P_2) \geq 4$. ■

Theorem 3.5. For $n \geq 5$, $n+1 \leq s(C_n \circ P_2) \leq 3n-3$.

Proof. For $n \geq 5$, consider the n^{th} self-amalgamation of $C_n \circ P_2$ defined by $*^n = (X, Y)$, with $X = \langle x_i \rangle_1^n = \langle a_1^1, a_1^2, a_1^3, \dots, a_1^{n-1}, a_1^n \rangle$ and $Y = \langle y_i \rangle_1^n = \langle a_2^2, a_2^3, a_2^4, \dots, a_2^n, a_2^1 \rangle$. Note that $d(1, a_1^3; (C_n \circ P_2)^{*^n}) = 3$, and $(C_n \circ P_2)^{*^n}$ is amalgamation-unstable for this choice of amalgamation sequences. Thus, $S^n(C_n \circ P_2)$ contains an amalgamation-unstable graph. Therefore, $s(C_n \circ P_2) \geq n+1$, $n \geq 5$. By Remark 1.3, $s(C_n \circ P_2) \leq 3n-3$. This completes the proof. ■

Example 3.1. Figure 3.5 shows a 5th self-amalgamation $(C_5 \circ P_2)^{*^5}$, with $*^5 = (\langle a_1^1, a_1^2, a_1^3, a_1^4, a_1^5 \rangle, \langle a_2^2, a_2^3, a_2^4, a_2^5, a_2^1 \rangle)$. Since a shortest path between 1 and a_1^3 is $(1, 5, a_1^4, a_1^3)$, which is a path of length 3, so $(C_5 \circ P_2)^{*^5}$ is amalgamation-unstable. Therefore, $s(C_5 \circ P_2) \geq 6$.

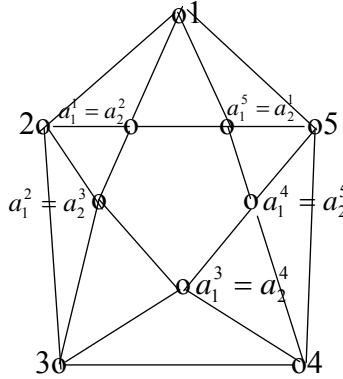


Figure 3.5 An amalgamation-unstable $(C_5 \circ P_2)^{*^5}$, with

$$*^5 = (\langle a_1^1, a_1^2, a_1^3, a_1^4, a_1^5 \rangle, \langle a_2^2, a_2^3, a_2^4, a_2^5, a_2^1 \rangle)$$

Remark 3.2. The n^{th} self-amalgamation of $C_n \circ P_2$ which is indicated in Theorem 3.5, defined by $*^n = (X, Y)$ with $X = \langle x_i \rangle_1^n = \langle a_1^1, a_1^2, a_1^3, \dots, a_1^{n-1}, a_1^n \rangle$ and

$Y = \langle y_i \rangle_1^n = \langle a_2^2, a_2^3, a_2^4, \dots, a_2^n, a_2^1 \rangle$, yields amalgamation-stable graphs when $n = 3$ or 4 instead. Thus, $*(C_3 \circ P_2) \leq 3$ and $*(C_4 \circ P_2) \leq 4$. In fact, $*(C_3 \circ P_2) = 2$ by Corollary 3.2, and $*(C_4 \circ P_2) = 3$ by Remark 3.1.

Theorem 3.6. $s(C_n \circ P_m) \geq n+1$, $n \geq 3$, $m \geq 3$

Proof. Let $n \geq 3$ and $m \geq 3$. Let $C_n = [1, 2, 3, \dots, n]$, $P_m = (a_1, a_2, a_3, \dots, a_m)$, and let the i^{th} copy of P_m be denoted by $P_m^i = (a_1^i, a_2^i, a_3^i, \dots, a_m^i)$, with each vertex of P_m^i joined by an edge to vertex i of the cycle C_n . Consider the n^{th} self-amalgamation of $C_n \circ P_m$ defined by $*^n = (X, Y)$, with $X = \langle x_i \rangle_1^n = \langle a_1^1, a_1^2, a_1^3, \dots, a_1^{n-1}, a_1^n \rangle$ and $Y = \langle y_i \rangle_1^n = \langle a_m^2, a_m^3, a_m^4, \dots, a_m^n, a_m^1 \rangle$. Note that $d(a_1^1, a_2^3; (C_n \circ P_m)^{*^n}) = 3$, and hence $(C_n \circ P_m)^{*^n}$ is amalgamation-unstable. Thus, $S^n(C_n \circ P_m)$ contains an amalgamation-unstable graph. Therefore, $s(C_n \circ P_m) \geq n+1$. ■

Example 3.2. An amalgamation-unstable 4th self-amalgamation of $C_4 \circ P_4$, with $*^4 = (\langle a_1^1, a_1^2, a_1^3, a_1^4 \rangle, \langle a_4^2, a_4^3, a_4^4, a_4^1 \rangle)$ is shown in Figure 3.6. Since a shortest path between a_1^1 and a_2^3 is $(a_1^1, 2, 3, a_2^3)$, so $C_4 \circ P_4$ is amalgamation-unstable. Therefore, $s((C_4 \circ P_4)^{*^4}) \geq 5$.

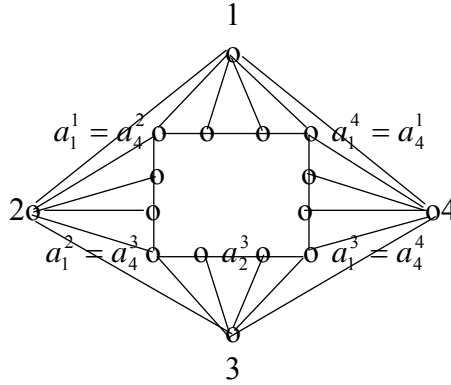


Figure 3.6 An amalgamation unstable $(C_4 \circ P_4)^{*^4}$,
with $*^4 = (\langle a_1^1, a_1^2, a_1^3, a_1^4 \rangle, \langle a_4^2, a_4^3, a_4^4, a_4^1 \rangle)$

The stability number and self-amalgamation number for certain coronas and crowns were established. However, sharper bounds may be obtained for the general cases.

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